CONTINUOUS GROUP ACTIONS ON PROFINITE SPACES

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ABSTRACT. For a profinite group, we construct a model structure on profinite spaces and profinite spectra with a continuous action. This yields descent spectral sequences for the homotopy groups of homotopy fixed point space and for stable homotopy groups of homotopy orbit spaces. Our main example is the Galois action on profinite étale topological types of varieties over a field. One motivation is to understand Grothendieck's section conjecture in terms of homotopy fixed points.

1. INTRODUCTION

Let $\hat{\mathcal{S}}$ be the category of profinite spaces, i.e simplicial objects in the category of profinite sets. Examples of profinite spaces arise in algebraic geometry. For a locally noetherian scheme X, we denote by $\hat{\mathrm{Et}}\, X$ the profinitely completed étale topological type of X of Friedlander, see [11] and [30]. It is a profinite space that collects the information of the étale topology on the scheme. Now let $\bar{k}$ be a separable closure of k and G_k its Galois group over k. Let X be a variety over k and let $\bar{X} = X \otimes_k \bar{k}$ be the base change of X to $\bar{k}$. Since G_k acts on $\bar{X}$ and since $\hat{\mathrm{Et}}$ is a functor, there is an induced G_k-action on $\hat{\mathrm{Et}}\, \bar{X}$. This Galois action is of course an important property of the étale topological type of $\bar{X}$. Furthermore, there is a natural sequence in $\hat{\mathcal{S}}$

$$\hat{\mathrm{Et}}\, \bar{X} \longrightarrow \hat{\mathrm{Et}}\, X \longrightarrow \hat{\mathrm{Et}}\, k. \tag{1}$$

As $\hat{\mathrm{Et}}\, k \simeq BG_k$, this inspires to think of $\hat{\mathrm{Et}}\, X$ as the homotopy orbit of $\hat{\mathrm{Et}}\, \bar{X}$ under its G_k-action. In fact, this would generalize a theorem of Cox' on real algebraic varieties in [5] that there is a weak equivalence of pro-spaces $\mathrm{Et}\, X \simeq X(\mathbb{C}) \times_G EG$ for $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

The main purpose of this paper is to provide a rigid framework for the Galois action on étale topological types via model categories in which a generalization of Cox' result and its applications can be proven. In particular, we are going to construct homotopy fixed points of étale topological types of varieties over arbitrary base fields.

So let us describe the general setup and thereby outline the content of the paper. Let G be an arbitrary profinite group and let $\hat{\mathcal{S}}_G$ be the category of profinite spaces with G acting continuously in each level and equivariant face and degeneracy maps. Various model structures on $\hat{\mathcal{S}}$ have been constructed, see [25] and [30]. We construct a left proper fibrantly generated model structure on $\hat{\mathcal{S}}_G$ such that the weak equivalences (cofibrations) in $\hat{\mathcal{S}}_G$ are exactly the maps that are weak equivalences (resp. cofibrations) in $\hat{\mathcal{S}}$. The construction is inspired by the one of Goerss in [12]

for profinite groups acting on simplicial discrete sets. The subtle point is that the mapping space functor $\hom_{\hat{\mathcal{S}}}(G, -) : \hat{\mathcal{S}} \to \hat{\mathcal{S}}_G$ is not the natural right adjoint of the forgetful functor $\hat{\mathcal{S}} \to \hat{\mathcal{S}}_G$ since the sets $\mathrm{Hom}_{\hat{\mathcal{S}}}(X, Y)$ do not have to be profinite for general X and Y in $\hat{\mathcal{S}}$. This problem can be circumvented by an intermediate model structure and a localization argument.

Let EG denote the universal profinite covering space of the classifying space BG. A convenient point of profinite spaces is that EG and BG are objects of $\hat{\mathcal{S}}$. Hence the homotopy fixed points of a profinite G-space X can be defined as the simplicial mapping space $X^{hG} = \hom_G(EG, RX)$ of continuous G-equivariant maps, where RX denotes a functorial fibrant replacement in $\hat{\mathcal{S}}_G$. Then we construct a descent spectral sequence for homotopy groups of homotopy fixed points of connected pointed profinite G-spaces

$$E_2^{s,t} = H^s(G; \pi_t X) \Rightarrow \pi_{t-s}(X^{hG}).$$

The E_2-term of this spectral sequence is continuous cohomology of the profinite groups $\pi_t X$, where $\pi_1 X$ might be a nonabelian profinite group.

For the homotopy orbit space $X_{hG} = X \times_G EG$, we construct a spectral sequence computing the homology $H_*(X_{hG}; M)$ for any profinite abelian group M. Both for homotopy fixed points and homotopy orbit spaces, the construction of the spectral sequence follows naturally from the work of Bousfield-Kan [3].

There is also a stable homotopy category $\hat{\mathcal{SH}}_G$ for G-spaces using profinite G-spectra. The well known machinery yields a homotopy orbit spectral sequence for stable profinite homotopy groups. This generalizes the notion of pro-f-spectra of Davis [6].

In [32], we study homotopy fixed points of profinite spectra with a continuous G-action in more detail. The main application is to provide a natural setting for the continuous action of the extended Morava stabilizer group G_n on Lubin-Tate spectra E_n. Since G_n acts continuously on the profinite homotopy groups $\pi_k E_n$, it seems natural to study the spectra E_n as profinite spectra. The construction of a descent spectral sequence of [8] for the homotopy fixed point spectra $E_n^{hG_n}$, respectively E_n^{hG} for any closed subgroup of G_n, then follows easily in the category of profinite spectra. These methods provide, in particular, a new construction for homotopy fixed points under open subgroups of G_n.

In the last section we return to the situation of a Galois group G_k acting on the variety $\bar{X}$. We will prove the following generalization of Cox's theorem mentioned above.

Theorem 1.1. *Let k be a field with absolute Galois group G_k and let X be a geometrically connected variety over k. Then the canonical map*

$$(\lim_L \hat{\mathrm{Et}}\, X_L) \times_{G_k} EG_k \to \hat{\mathrm{Et}}\, X$$

is a weak equivalence of profinite spaces, where the limit is taken over all finite Galois extensions L/k in $\bar{k}$ and X_L denotes $X \otimes_k L$.

We would like to prove the theorem directly for $\hat{\mathrm{Et}}\, \bar{X}$ but it is not clear that $\hat{\mathrm{Et}}\, \bar{X}$ is an object of the category of profinite G_k-spaces defined above. But the canonical map G_k-equivariant map $\hat{\mathrm{Et}}\, \bar{X} \longrightarrow \lim_L \hat{\mathrm{Et}}\, X_L$ is a weak equivalence of profinite spaces and the latter space has all the properties we need. Nevertheless, for a

variety over a field, the theorem provides the following intuition with a precise meaning: The two perspectives of viewing the étale topological type of X as a profinite space $\hat{\mathrm{Et}}\, X$ over $\hat{\mathrm{Et}}\, k \simeq BG_k$ or as a profinite space $\hat{\mathrm{Et}}\, \bar{X}$ together with its induced G_k-action are essentially equivalent.

By Theorem 1.1, the homotopy orbit spectral sequence above may be written as a Galois-descent spectral sequence for stable profinite étale homotopy groups of X:

$$E_{p,q}^2 = H_p(G_k; \pi_q^{\text{ét},s}(\bar{X})) \Rightarrow \pi_{p+q}^{\text{ét},s}(X).$$

Moreover, we will show that the stable étale realization functor can be viewed as a functor from motivic spectra to $\hat{\mathcal{SH}}_{G_k}$. Finally, we show that a refined version of étale cobordism of [29] satisfies Galois descent in the sense that for any variety X over k there is a spectral sequence

$$E_2^{s,t} = H^s(G_k; \hat{M}U_{\text{ét}}^t(\bar{X})) \Rightarrow \hat{M}U_{\text{ét}}^{s+t}(X).$$

An ℓ-adic version of étale (co)bordism has been used in [31] in order to study the integral cycle map from algebraic cycles to étale homology for schemes over an algebraically closed base field. The descent spectral sequence above should be useful for a future application of the techniques of [31] for varieties over a finite field.

But the main motivation for studying homotopy fixed points under the Galois action is Grothendieck's section conjecture. In fact, the conjecture can be formulated as an isomorphism on k-rational points $X(k)$ and G_k-homotopy fixed points of $\hat{\mathrm{Et}}\, \bar{X}$. Namely, as the curves involved in the conjecture are $K(\pi, 1)$-varieties, we get an isomorphism for continuous cohomology

(2) $$H^1(G_k; \pi_1 \bar{X}) \cong \pi_0(B\pi_1 \bar{X})^{hG_k} \cong \pi_0(\hat{\mathrm{Et}}\, \bar{X})^{hG_k}.$$

So one could read the section conjecture as the conjecture that the canonical map from $X(k)$ to the homotopy fixed point sets on the right hand side of (2) is bijective. This point of view might be of interest, as analogues of the section conjecture over the reals, first proven by Mochizuki in [24], could be proved by Pal using homotopy fixed points results [27]. Over $\mathbb{R}$, Cox' theorem is a crucial point in the proof of the section conjecture. It is likely, that its generalization, Theorem 1.1 above, should be useful for an extension of the methods over $\mathbb{R}$ to fields finitely generated over $\mathbb{Q}$ or $\mathbb{Q}_p$.

The key new progress of this paper for this direction is that, since $B\pi_1 \bar{X}$ and $\hat{\mathrm{Et}}\, \bar{X}$ are naturally profinite spaces, the machinery described above provides a good notion of homotopy fixed points of these spaces which reflects the fact that the left hand side of (2) is continuous cohomology with profinite coefficients. This opens a new homotopy theoretical tool kit to analyze the section conjecture.

2. Homotopy theory of profinite G-spaces

2.1. **Profinite spaces.** First, we recall some basic notions for profinite spaces and their homotopy category from [25] and [30]. For a category $\mathcal{C}$ with small limits, the pro-category of $\mathcal{C}$, denoted pro-$\mathcal{C}$, has as objects all cofiltering diagrams $X : I \to \mathcal{C}$. Its sets of morphisms are defined as

$$\mathrm{Hom}_{\mathrm{pro}-\mathcal{C}}(X, Y) := \lim_{j \in J} \mathrm{colim}_{i \in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j).$$

A constant pro-object is indexed by the category with one object and one identity map. The functor sending an object X of $\mathcal{C}$ to the constant pro-object with value X makes $\mathcal{C}$ a full subcategory of pro-$\mathcal{C}$. The right adjoint of this embedding is the limit functor lim: pro-$\mathcal{C} \to \mathcal{C}$, which sends a pro-object X to the limit in $\mathcal{C}$ of the diagram corresponding to X.

Let $\mathcal{E}$ denote the category of sets and let $\mathcal{F}$ be the full subcategory of finite sets. Let $\hat{\mathcal{E}}$ be the category of compact Hausdorff totally disconnected topological spaces. We may identify $\mathcal{F}$ with a full subcategory of $\hat{\mathcal{E}}$ in the obvious way. The limit functor lim: pro-$\mathcal{F} \to \hat{\mathcal{E}}$ is an equivalence of categories.

We denote by $\hat{\mathcal{S}}$ (resp. $\mathcal{S}$) the category of simplicial profinite sets (resp. simplicial sets). The objects of $\hat{\mathcal{S}}$ (resp. $\mathcal{S}$) will be called *profinite spaces* (resp. *spaces*). The forgetful functor $\hat{\mathcal{E}} \to \mathcal{E}$ admits a left adjoint $(\hat{\cdot}) : \mathcal{E} \to \hat{\mathcal{E}}$. It induces a functor $(\hat{\cdot}) : \mathcal{S} \to \hat{\mathcal{S}}$, which is called *profinite completion*. It is left adjoint to the forgetful functor $|\cdot| : \hat{\mathcal{S}} \to \mathcal{S}$ which sends a profinite space to its underlying simplicial set.

For a profinite space X we define the set $\mathcal{R}(X)$ of simplicial open equivalence relations on X. An element R of $\mathcal{R}(X)$ is a simplicial profinite subset of the product $X \times X$ such that, in each degree n, R_n is an equivalence relation on X_n and an open subset of $X_n \times X_n$. It is ordered by inclusion. For every element R of $\mathcal{R}(X)$, the quotient X/R is a simplicial finite set and the map $X \to X/R$ is a map of profinite spaces. The canonical map $X \to \lim_{R \in \mathcal{R}(X)} X/R$ is an isomorphism in $\hat{\mathcal{S}}$, cf. [25], Lemme 1.

Let X be a profinite space. The continuous cohomology $H^*(X; \pi)$ of X with coefficients in the topological abelian group π is defined as the cohomology of the complex $C^*(X; \pi)$ of continuous cochains of X with values in π, i.e. $C^n(X; \pi)$ denotes the set $\mathrm{Hom}_{\hat{\mathcal{E}}}(X_n, \pi)$ of continuous maps $\alpha : X_n \to \pi$ and the differentials $\delta^n : C^n(X; \pi) \to C^{n+1}(X; \pi)$ are the morphisms associating to α the map $\sum_{i=0}^{n+1} \alpha \circ d_i$, where d_i denotes the ith face map of X, see [36] and [25]. If π is a finite abelian group and Z a simplicial set, then the cohomologies $H^*(Z, \pi)$ and $H^*(\hat{Z}, \pi)$ are canonically isomorphic.

If π is an arbitrary profinite group, we may still define the first cohomology of X with coefficients in π as done by Morel in [25], p. 355. The functor

$$X \mapsto \mathrm{Hom}_{\hat{\mathcal{E}}}(X_0, \pi)$$

is represented in $\hat{\mathcal{S}}$ by a profinite space $E\pi$. We define the 1-cocycles $Z^1(X; \pi)$ to be the set of continuous maps $f : X_1 \to \pi$ such that $f(d_0 x) f(d_2 x) = f(d_1 x)$ for every $x \in X_1$. The functor $X \mapsto Z^1(X; \pi)$ is represented by a profinite space $B\pi$. Explicit constructions of $E\pi$ and $B\pi$ may be given in the standard way as in $\mathcal{S}$. Furthermore, there is a map $\delta : \mathrm{Hom}_{\hat{\mathcal{S}}}(X, E\pi) \to Z^1(X; \pi) \cong \mathrm{Hom}_{\hat{\mathcal{S}}}(X, B\pi)$ which sends $f : X_0 \to \pi$ to the 1-cocycle $x \mapsto \delta f(x) = f(d_0 x) f(d_1 x)^{-1}$. We denote by

$B^1(X;\pi)$ the image of δ in $Z^1(X;\pi)$ and we define the pointed set $H^1(X,\pi)$ to be the quotient $Z^1(X;\pi)/B^1(X;\pi)$. Finally, if X is a profinite space, we define $\pi_0 X$ to be the coequalizer in $\hat{\mathcal{E}}$ of the diagram $d_0, d_1 : X_1 \rightrightarrows X_0$.

The profinite fundamental group of X is defined via covering spaces. There is a universal profinite covering space $(\tilde{X}, x)$ of X at a vertex $x \in X_0$. Then $\pi_1(X, x)$ is defined to be the group of automorphisms of $(\tilde{X}, x)$ over (X, x). It has a natural structure of a profinite group as the limit of the finite automorphism groups of the finite Galois coverings of (X, x). For any details, we refer the reader to [30]. Its relation to the usual fundamental group of a simplicial set is described by the following result.

Proposition 2.1. *For a pointed simplicial set X, the canonical map from the profinite group completion of $\pi_1(X)$ to $\pi_1(\hat{X})$ is an isomorphism, i.e.*

$$\widehat{\pi_1(X)} \cong \pi_1(\hat{X})$$

as profinite groups.

Now we are able to define the weak equivalences in $\hat{\mathcal{S}}$.

Definition 2.2. *A morphism $f : X \to Y$ in $\hat{\mathcal{S}}$ is called*
1) a weak equivalence if the induced map $f_* : \pi_0(X) \to \pi_0(Y)$ is an isomorphism of profinite sets, $f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$ is an isomorphism of profinite groups for every vertex $x \in X_0$ and $f^* : H^q(Y, \mathcal{M}) \to H^q(X, f^*\mathcal{M})$ is an isomorphism for every local coefficient system $\mathcal{M}$ of finite abelian groups on Y for every $q \geq 0$;
2) a cofibration if f is a level-wise monomorphism;
3) a fibration if it has the right lifting property with respect to every cofibration that is also a weak equivalence, called trivial cofibrations.

This class of weak equivalences fits into a simplicial fibrantly generated left proper model structure on $\hat{\mathcal{S}}$. For every natural number $n \geq 0$ we choose a finite set with n elements, e.g. the set $\{0, 1, \ldots, n-1\}$, as a representative of the isomorphism class of sets with n elements. We denote the set of these representatives by $\mathcal{T}$. Moreover, for every isomorphism class of finite groups, we choose a representative with underlying set $\{0, 1, \ldots, n-1\}$. Hence for each n we have chosen as many groups as there are relations on the set $\{0, 1, \ldots, n-1\}$. This ensures that the collection of these representatives forms a set which we denote by $\mathcal{G}$.

Let P and Q be the following two sets of morphisms:

P	consisting of	$E\Gamma \to B\Gamma, B\Gamma \to *, L(M, n) \to K(M, n+1),$
		$K(M, n) \to *, K(S, 0) \to *$
		for every finite set $S \in \mathcal{T}$, every finite group $\Gamma \in \mathcal{G}$,
		every finite abelian group $M \in \mathcal{G}$ and every $n \geq 0$;
Q	consisting of	$E\Gamma \to *, L(M, n) \to *$ for every finite group $\Gamma \in \mathcal{G}$,
		every finite abelian group $M \in \mathcal{G}$ and every $n \geq 0$.

The following theorem has been proved in [30].

Theorem 2.3. *The above defined classes of weak equivalences, cofibrations and fibrations provide $\hat{\mathcal{S}}$ with the structure of a fibrantly generated left proper model category with P the set of generating fibrations and Q the set of generating trivial fibrations. We denote the homotopy category by $\hat{\mathcal{H}}$.*

We consider the category $\mathcal{S}$ of simplicial sets with the model structure of [33] and denote its homotopy category by $\mathcal{H}$. For the proof of the following proposition, we refer again to [30].

Proposition 2.4. *1. The level-wise completion functor $\hat{(\cdot)} : \mathcal{S} \to \hat{\mathcal{S}}$ preserves weak equivalences and cofibrations.*
2. The forgetful functor $|\cdot| : \hat{\mathcal{S}} \to \mathcal{S}$ preserves fibrations and weak equivalences between fibrant objects.
3. The induced completion functor $\hat{(\cdot)} : \mathcal{H} \to \hat{\mathcal{H}}$ and the right derived functor $R|\cdot| : \hat{\mathcal{H}} \to \mathcal{H}$ form a pair of adjoint functors.

Definition 2.5. *Let X be a pointed profinite space and let RX be a fibrant replacement of X in the above model structure on $\hat{\mathcal{S}}_*$. Then we define the nth profinite homotopy group of X for $n \geq 2$ to be the profinite group*

$$\pi_n(X) := \pi_0(\Omega^n(RX)).$$

One should note, that to be a fibration in $\hat{\mathcal{S}}$ is a stronger condition than in $\mathcal{S}$. The profinite structure of the $\pi_n X$, would not be obtained by taking homotopy groups for $|X| \in \mathcal{S}$.

Remark 2.6. Morel [25] proved that there is a model structure on $\hat{\mathcal{S}}$ for each prime number p in which the weak equivalences are maps that induce isomorphisms on $\mathbb{Z}/p$-cohomology. The fibrant replacement functor R^p yields a rigid version of Bousfield-Kan $\mathbb{Z}/p$-completion. The homotopy groups for this structure are pro-p-groups being defined in the same way as above using R^p instead of R.

2.2. **Profinite G-spaces.** Let G be a fixed profinite group. Let X be a profinite set on which G acts continuously, i.e. G acts on X and the action map $\mu : G \times X \to X$ is continuous. In this situation we say that X is a profinite G-set.
If X is a profinite space and G acts continuously on each X_n such that the action is compatible with the structure maps, then we call X a profinite G-space. We denote by $\hat{\mathcal{S}}_G$ the category of profinite G-spaces with level-wise continuous G-equivariant morphisms. For an open and hence closed normal subgroup U of G, let X_U be the quotient space under the action by U, i.e. the quotient $X/\sim$ with $x \sim y$ in X if both are in the same orbit under U. The following lemma is the analogue of the characterization of discrete spaces with a profinite group action.

Lemma 2.7. *Let G be a profinite group and X a profinite space. Then X is a profinite G-space if and only if the canonical map $\phi : X \to \lim_U X_U$ is an isomorphism, where U runs through the open normal subgroups of G.*

Remark 2.8. Let us quickly recall the construction of colimits in $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}_G$. If $\{X_i\}_{i \in I}$ is a diagram of profinite spaces, one can construct its colimit in $\hat{\mathcal{S}}$ as follows. Let X be the colimit of the underlying diagram of spaces, i.e. $X := \text{colim}_i |X_i|$, and let $\varphi_i : X_i \to X$ be the canonical maps in $\mathcal{S}$. We define a set $\mathcal{R}$ of equivalence relations on X, which are simplicial subsets of $X \times X$, to be the set of simplicial equivalence relations R on X such that
- X_n/R_n is finite in each degree n, i.e. X/R is a simplicial finite set,
- $\varphi_i^{-1}(R)$ is open in each $X_i \times X_i$ for all i, i.e. $\varphi_i^{-1}(R_n)$ is an open subset in each $X_{i,n} \times X_{i,n}$.

Then $\mathcal{R}$ is filtered from below and we define the colimit of the diagram in $\hat{\mathcal{S}}$ to be the completion of X with respect to $\mathcal{R}$, i.e. $\hat{X} := \lim_{R \in \mathcal{R}} X/R$ in $\hat{\mathcal{S}}$. It is equipped with a canonical map $X \xrightarrow{\iota} \hat{X}$ which sends each $x \in X$ to the sequence of its equivalence classes $[x]_R \in X/R$. The image of ι is dense in $\hat{X}$. There are canonical maps $\hat{\varphi}_i : X_i \xrightarrow{\varphi_i} X \xrightarrow{\iota} \hat{X}$ in $\hat{\mathcal{S}}$ that provide $\hat{X}$ with the universal property of a colimit in $\hat{\mathcal{S}}$.

If $\{X_i\}_{i \in I}$ is a diagram of profinite G-spaces, then we modify $\mathcal{R}$ to $\mathcal{R}_G$ by the additional condition that every $R \in \mathcal{R}_G$ is in addition a G-invariant subspace of $X \times X$. Then we define the colimit of the diagram to be the completion of the underlying colimit with respect to $\mathcal{R}_G$.

For X and Y in $\hat{\mathcal{S}}$, the simplicial mapping space $\hom(X, Y)$ is defined in degree n as the set of continuous maps $\mathrm{Hom}_{\hat{\mathcal{S}}}(X \times \Delta[n], Y)$. If G is a finite discrete group, considered as a constant simplicial profinite set, $\hom(G, Y)$ has a natural profinite structure induced by the profinite structure on Y. In order to show that $\hat{\mathcal{S}}_G$ is a model category, we would like to use a right adjoint functor to the forgetful functor $\hat{\mathcal{S}}_G \to \hat{\mathcal{S}}$. But the problem is, that if G is an arbitrary profinite group, the natural candidate for the right adjoint $\hom(G, Y)$ does not have to be a profinite space. This forces us to consider an intermediate structure as in [12].

Let U be an open normal subgroup of G and let $F_{G/U} : \hat{\mathcal{S}}_G \to \hat{\mathcal{S}}$, denote the composition of the functor $\hat{\mathcal{S}}_G \to \hat{\mathcal{S}}_{G/U}$, $X \mapsto X_U$, followed by the forgetful functor. We denote by $\hom(G/U, -)$ the functor $\hat{\mathcal{S}} \to \hat{\mathcal{S}}_G$, $Y \mapsto \hom(G/U, Y)$.

Lemma 2.9. *For each open normal subgroup U of G, the functor $\hom(G/U, -)$ is right adjoint to $F_{G/U}$.*

Proof. A G-equivariant map $X \to \hom(G/U, Y)$ factors through X_U, since U acts trivially on $\hom(G/U, Y)$. $\square$

Definition 2.10. *A map $f : X \to Y$ in $\hat{\mathcal{S}}_G$ is called a* strict weak equivalence *(resp.* strict cofibration*) if $f_U : X_U \to Y_U$ is a weak equivalence (resp. cofibration) in $\hat{\mathcal{S}}$ for every open normal subgroup U of G. It is called a* strict fibration *if it has the right lifting property with respect to strict trivial cofibrations.*

Lemma 2.11. *Let $i : A \to B$ be a map in $\hat{\mathcal{S}}_G$. Then i is a strict cofibration in $\hat{\mathcal{S}}_G$ if and only if i has the left lifting property with respect to the maps $\hom(G/U, q)$ for every map q of Q and open normal subgroups U of G.*

Proof. This follows from the adjointness of Lemma 2.9 and Theorem 2.12 of [30]. $\square$

The same proof shows the analogue statement for strict trivial cofibrations.

Lemma 2.12. *Let $i : A \to B$ be a map in $\hat{\mathcal{S}}_G$. Then i is a strict trivial cofibration in $\hat{\mathcal{S}}_G$ if and only if i has the left lifting property with respect to the maps $\hom(G/U, p)$ for every map p of P and open normal subgroups U of G.*

We refer the reader for the following notions of cosmallness and the cosmall object argument, which is the dual of the small object argument, to [16] and [4]. Furthermore, we write $\hom(G/U, P)$ (respectively $\hom(G/U, Q)$) for the set of maps

$\mathrm{hom}(G/U, p)$ (respectively $\mathrm{hom}(G/U, q)$) in $\hat{\mathcal{S}}_G$ such that p lies in P (respectively q lies in Q).

Lemma 2.13. *The sets* $\mathrm{hom}(G/U, P)$ *and* $\mathrm{hom}(G/U, Q)$ *permit the cosmall object argument.*

Proof. We have to show that the codomains of the set $\mathrm{hom}(G/U, P)$ (respectively $\mathrm{hom}(G/U, Q)$) are cosmall relative to $\mathrm{hom}(G/U, P)$ (respectively $\mathrm{hom}(G/U, Q)$). The only non-trivial spaces among those are $\mathrm{hom}(G/U, B\Gamma)$ and $\mathrm{hom}(G/U, K(M, n))$, for $n \geq 1$. Since G/U is a constant simplicial profinite set, a map $G/U \to Y$ in $\hat{\mathcal{S}}$ is completely determined by its 0-component $G/U \to Y_0$ in $\hat{\mathcal{E}}$. Moreover, a map $\Delta[n] \to Y$ corresponds uniquely to an n-simplex of Y. Since the spaces $B\Gamma$ and $K(M, n)$, for $n \geq 1$, have only one 0-simplex, $\mathrm{hom}(G/U, B\Gamma)$ (resp. $\mathrm{hom}(G/U, K(M, n))$) is isomorphic to $B\Gamma$ (resp. $K(M, n)$). So the cosmallness follows as in the proof of Theorem 2.12 of [30]. $\qquad\square$

Lemma 2.14. *Every map* $f : X \to Y$ *in* $\hat{\mathcal{S}}_G$ *can be factored into*

$$X \xrightarrow{\ j\ } Z \xrightarrow{\ q\ } Y$$

where j is a strict cofibration and q is a strict trivial fibration.

Proof. We construct Z using the cosmall object argument. Let λ be a regular cardinal such that every codomain of $\mathrm{hom}(G/U, Q)$ is λ-cosmall relative to relative $\mathrm{hom}(G/U, Q)$-cocell complexes, where the $\mathrm{hom}(G/U, Q)$-cocell complexes are maps that are transfinite compositions of pullbacks of elements of $\mathrm{hom}(G/U, Q)$. We set $Z_0 = Y$ and define $Z_{\beta+1}$ for inductively for $\beta < \lambda$ as the pullback of the diagram

$$
\begin{array}{ccc}
Z_{\beta+1} & \longrightarrow & \prod_{d \in D} \mathrm{hom}(G/U_d, R_d) \\
\downarrow & & \downarrow {\scriptstyle \mathrm{hom}(G/U_d, q_d)} \\
Z_\beta & \longrightarrow & \prod_{d \in D} \mathrm{hom}(G/U_d, S_d)
\end{array}
$$

where $\mathrm{hom}(G/U_d, q_d) : \mathrm{hom}(G/U_d, R_d) \to \mathrm{hom}(G/U_d, S_d)$ is a map in $\mathrm{hom}(G/U, Q)$ and D denotes the set of all diagrams

$$
\begin{array}{ccc}
X & \longrightarrow & \mathrm{hom}(G/U, R) \\
\downarrow & & \downarrow {\scriptstyle \mathrm{hom}(G/U, q)} \\
Z_\beta & \longrightarrow & \mathrm{hom}(G/U, S).
\end{array}
$$

This yields a factorization $X \xrightarrow{\ j\ } Z \xrightarrow{\ q\ } Y$ such that j has the left lifting property with respect to $\mathrm{hom}(G/U, Q)$ and is a strict cofibration and q has the right lifting property with respect to all strict cofibrations by Lemma 2.11. By [33] II §2, this implies that it is a simplicial homotopy equivalence, hence also a weak equivalence and a strict fibration. $\qquad\square$

Lemma 2.15. *Every map* $f : X \to Y$ *in* $\hat{\mathcal{S}}_G$ *can be factored into*

$$X \xrightarrow{\ i\ } Z \xrightarrow{\ p\ } Y$$

where i is a strict trivial cofibration and p is a strict fibration.

Proof. This follows from the cosmall object argument using Lemma 2.12. □

We denote by $\hom(G/U, P)$-proj the maps having the left lifting property with respect to all maps in $\hom(G/U, P)$; and by $\hom(G/U, P)$-fib the maps having the right lifting property with respect to all maps in $\hom(G/U, P)$-proj.

Lemma 2.16. *The strict fibrations in $\hat{\mathcal{S}}_G$ are exactly the maps $\hom(G/U, P)$-fib for all open normal subgroups U of G.*

Proof. By Lemma 2.15, any map $f : X \to Y$ can be factored as $X \xrightarrow{i} Z \xrightarrow{p} Y$ such that i is a strict trivial cofibration and p is a relative $\hom(G/U, P)$-cocell complex. If f is a strict fibration, then f is a retract of p and is hence in $\hom(G/U, P)$-fib. □

Lemma 2.17. *A map $i : A \to B$ in $\hat{\mathcal{S}}_G$ is a strict cofibration if and only if it is a level-wise injection.*

Proof. The map i is a strict fibration if all i/U are injective. Hence their limit is level-wise injective. Conversely, if i is an injective G-equivariant map, then all quotient maps i/U are obviously injective. □

We define two sets of maps $\tilde{P}$ consisting of all maps $\hom(G/U, p)$ for every $p \in P$ and every open normal subgroup $U \subseteq G$ and $\tilde{Q}$ consisting of all maps $\hom(G/U, q)$ for every $q \in Q$ and every open normal subgroup $U \subseteq G$. The next result now follows immediately from the previous arguments.

Theorem 2.18. *The strict weak equivalences, strict cofibrations and strict fibrations give $\hat{\mathcal{S}}_G$ the structure of a fibrantly generated left proper simplicial model category with $\tilde{P}$ the set of generating fibrations and $\tilde{Q}$ the set of generating trivial fibrations.*

Finally we enlarge the class of weak equivalences. We say that a morphism in $\hat{\mathcal{S}}_G$ is

- a weak equivalence if it is a weak equivalence in $\hat{\mathcal{S}}$;
- a cofibration if it is a level-wise monomorphism;
- a fibration if it has the right lifting property with respect to all trivial cofibrations.

By Proposition 2.23 of [30], the limit functor in $\hat{\mathcal{S}}$ is homotopy invariant and hence every strict weak equivalence is a weak equivalence in the above sense.

Theorem 2.19. *These classes of maps define the structure of a left proper fibrantly generated simplicial model category on the category of profinite G-spaces. We denote its homotopy category by $\hat{\mathcal{H}}_G$. The underlying map of a fibration in $\hat{\mathcal{S}}_G$ is also a fibration in $\hat{\mathcal{S}}$ (and in $\mathcal{S}$).*

Proof. This new model structure is obtained as the left Bousfield localization of the previous strict structure. From every isomorphism class of objects in $\hat{\mathcal{S}}_G$ which are fibrant in $\hat{\mathcal{S}}$, we choose a representative. We let K be the set of these representing

objects in $\hat{\mathcal{S}}_G$. Then the K-local equivalences are exactly the maps in $\hat{\mathcal{S}}_G$ that are weak equivalences in $\hat{\mathcal{S}}$. The result now follows from Theorem 6 of [29], which is a modification of the localization theorems of [16]. The last statement follows directly from Proposition 2.4. □

Definition 2.20. *Let X be a profinite G-space and M a profinite G-module. We define the G-equivariant cohomology of X with coefficients in M to be*

$$H_G^n(X, M) := \mathrm{Hom}_{\hat{\mathcal{H}}_G}(X, K(M, n)).$$

Remark 2.21. Let p be any prime number. The method to prove Theorem 2.19 also applies to Morel's $\mathbb{Z}/p$-model structure on $\hat{\mathcal{S}}$ of [25] and the action of a profinite group G. The set of generating fibrations for this model structure is the set of canonical maps $L(\mathbb{Z}/p, n) \to K(\mathbb{Z}/p, n+1)$, $K(\mathbb{Z}/p, n) \to *$ for all $n \geq 0$; the set of generating trivial fibrations is the set of maps $L(\mathbb{Z}/p, n) \to *$ for every $n \geq 0$, see [30].

2.3. **Homotopy fixed points and homotopy orbits.** We define the homotopy fixed points as usually as the function space of maps coming from EG.

Definition 2.22. *Let G be a profinite group, let X be a profinite G-space and let $X \mapsto RX$ be a fixed functorial fibrant replacement in $\hat{\mathcal{S}}_G$. We define the profinite homotopy fixed point space of X to be the space of G-invariant maps from EG to RX:*

$$X^{hG} := \mathrm{hom}_G(EG, RX).$$

Let $\hat{\mathcal{S}}/BG$ denote the category of profinite spaces equipped with a map to BG with the model structure induced by the one on $\hat{\mathcal{S}}$ via the forgetful functor. There is a functor $\hat{\mathcal{S}}_G \to \hat{\mathcal{S}}/BG$ sending X to the Borel construction $X_{hG} := EG \times_G X \to BG$, which we call the homotopy orbit space of X. On the other hand, there is the functor $\hat{\mathcal{S}}/BG \to \hat{\mathcal{S}}_G$ sending $Y \to BG$ to the G-principal fibration $EG \times_{BG} Y$. Since the map $EG \times_{BG} X_{hG} = EG \times X \to X$ is a G-equivariant weak equivalence and $EG \times_G (EG \times_{BG} Y) \to Y$ is a weak equivalence of profinite spaces over BG, they induce a pair of adjoint functors between the homotopy categories. For a profinite G-module, we denote by $K^G(M, n)$ the profinite space $EG \times_G K(M, n)$.

Proposition 2.23. *Let M be a profinite G-module. Then the homotopy groups of the simplicial set $K(M, n)^{hG}$ satisfy are equal to the continuous cohomology of G, i.e. for $0 \leq k \leq n$ we have*

$$\pi_k K(M, n)^{hG} = H^{n-k}(G; M).$$

Proof. By definition of the cohomology $H^{n-k}(G; M)$ via homogeneous continuous cochains, there is an isomorphism $\pi_0 \mathrm{hom}_G(EG, K(M, n)) = H^n(G; M)$. The above adjointness induces an isomorphism $\pi_0 \mathrm{hom}_{\hat{\mathcal{S}}/BG}(BG, K^G(M, n)) = H^n(G; M)$. Now, applying the functor $\mathrm{hom}_{\hat{\mathcal{S}}/BG}(BG, -)$ to the homotopy fibre square

$$
\begin{array}{ccc}
K^G(M, n) & \longrightarrow & BG \\
\downarrow & & \downarrow \\
BG & \longrightarrow & K^G(M, n+1)
\end{array}
$$

shows that $\hom_{\hat{\mathcal{S}}/BG}(BG, K^G(M, n))$ is homotopy equivalent to the loop space $\Omega \hom_{\hat{\mathcal{S}}/BG}(BG, K^G(M, n+1))$. Hence $\pi_k \hom_{\hat{\mathcal{S}}/BG}(BG, K^G(M, n)) = H^{n-k}(G; M)$. □

One should note that, as indicated in the formulation of the proposition, there is a little subtlety about homotopy fixed points for profinite spaces. For an arbitrary profinite group G and a fibrant profinite space X, $X^{hG} = \hom_G(EG, X)$ is in general not a profinite space. Since the map $EG \to *$ is a trivial fibration between cofibrant objects, the induced map $\hat{X}^{hG} := \hom_G(*, X) \to X^{hG}$ is simplicial homotopy equivalence in $\mathcal{S}$. The space $\hat{X}^{hG}$ does have a natural profinite structure, but the homotopy groups of $\hom_G(*, X)$ and $\hom_G(EG, X)$ as simplicial sets are in general not profinite groups. In order to get a profinite homotopy type of $\hat{X}^{hG}$, one has to take again a fibrant replacement $R\hat{X}^{hG}$ of it in $\hat{\mathcal{S}}$. The fundamental group of $R\hat{X}^{hG}$ is then isomorphic to the profinite completion of $\pi_1 X^{hG}$ by Proposition 2.1. Nevertheless, X^{hG} is a useful object for studying actions of profinite groups. The situation is different for finitely generated profinite groups, for example in the case of a p-adic analytic profinite group G. So for a moment we suppose that G is a p-adic analytic group and that M is a profinite $\mathbb{Z}_p[[G]]$-module, being the inverse limit $M = \lim_\alpha M_\alpha$ of finite G-modules M_α. Let $X = K(M, n)$ be a fibrant Eilenberg MacLane space in $\hat{\mathcal{S}}$. By Proposition 2.23, we have $\pi_k K(M, n)^{hG} = H^{n-k}(G; M)$. Moreover, since G is p-adic analytic, it has an open normal subgroup which is a Poincaré pro-p-group. This implies that $H^n(G; M_\alpha)$ is finite for each α and that $H^n(G; M) = \lim_\alpha H^n(G; M_\alpha)$. This shows that in this case X^{hG} has itself a natural profinite structure and is fibrant in $\hat{\mathcal{S}}$.

Back to an arbitrary profinite group G, the fixed point functor $(-)^G$ is right adjoint to the functor from $\hat{\mathcal{S}} \to \hat{\mathcal{S}}_G$ sending X to itself as a trivial G-space. This functor clearly preserves weak equivalences and cofibrations, hence $(-)^G$ preserves weak equivalences between fibrant objects. Hence taking profinite homotopy fixed points $R\hat{X}^{hG}$ defines a functor $\hat{\mathcal{H}}_G \to \hat{\mathcal{H}}$ and may be viewed as the total right derived functor of $(-)^G$.

Moreover, a profinite G-space $X \in \hat{\mathcal{S}}_G$ may be considered as a functor from G as a groupoid to $\hat{\mathcal{S}}$. From this point of view, $\hom_G(*, X) = X^G$ is the limit of this functor in $\hat{\mathcal{S}}$. Moreover, for $X \in \hat{\mathcal{S}}_G$ fibrant, we consider $\hom_G(EG, X)$ as the homotopy limit in $\mathcal{S}$. One should note, that in [30] a profinite homotopy limit with values in $\hat{\mathcal{S}}$ has been considered. But Proposition 2.23and the following theorem show that, in the present case, $\hom_G(EG, X) \in \mathcal{S}$ is the object that we need here.

Theorem 2.24. *Let G be a profinite group and let X be a pointed profinite G-space. Assume either that G has finite cohomological dimension or that X has only finitely many nonzero homotopy groups. Then there is a strongly convergent descent spectral sequence for the homotopy groups of the homotopy fixed point space starting from continuous cohomology with profinite coefficients:*

$$E_2^{s,t} = H^s(G; \pi_t(X)) \Rightarrow \pi_{t-s}(X^{hG}).$$

Proof. This is a version of the homotopy limit spectral sequence of Bousfield and Kan for profinite spaces. We consider the category $c\hat{\mathcal{S}}$ of cosimplicial profinite spaces equipped with the model structure of [3] X, §4. As remarked in [3] XI, 5.7,

there is a cosimplicial replacement functor $\Pi^* X \in c\hat{\mathcal{S}}$ for a diagram of profinite spaces since there exist products in $\hat{\mathcal{S}}$. It is given in codimension n by $\Pi^n X = \hom_G(G^n, X) \in \hat{\mathcal{S}}$. If X is fibrant in $\hat{\mathcal{S}}_G$, its cosimplicial resolution is a fibrant object in $c\hat{\mathcal{S}}$. Now define the total profinite space of a cosimplicial profinite space Y to be

$$\mathrm{Tot}Y := \lim_s \mathrm{Tot}_s Y$$

where $\mathrm{Tot}_s Y := \hom(\Delta^{[s]}, Y)$, $\Delta[s]$ is the s-skeleton of the cosimplicial standard simplex and hom denotes the profinite continuous function space. Then there is a spectral sequence of the cosimplicial replacement of X which is the spectral sequence associated to the tower of fibrations that arises from the total profinite space of the cosimplicial replacement of X. We have to check that the E_2-term is continuous cohomology of G.

By an analogue of [3] X, 7.2, there are natural isomorphisms $E_2^{s,t} \cong \pi^s \pi_t(\Pi^* X)$ for $t \geq s \geq 0$, where π^s denotes the cohomotopy of the cosimplicial profinite group $\pi_t(\Pi^* X)$. Since $\Pi^n X$ is fibrant, there are natural isomorphisms $\pi_t \Pi^* X \cong \Pi^* \pi_t X$ by [3] XI, 5.7. This implies that the above cohomotopy are cohomology groups of the complex $C^*(G; \pi_t X)$ given in degree s by the set of continuous maps from $G^s \to \pi_t X$. If $\pi_1 X$ is not abelian, this also holds for $s = 0, 1$, where $H^s(G; \pi_1 X)$ is still a pointed set. Hence we have identified the E_2-term with the continuous cohomology groups of the statement.

It follows from the definition of $\Pi^* X$ that the total space of this cosimplicial object is equal to $\hom_G(EG, X)$, i.e. the abutment of the spectral sequence is $\pi_{t-s} X^{hG}$. Finally, the assumptions imply $\lim_r^1 E_r^{s,t}$ vanishes and the spectral sequence is strongly convergent. □

We recall from [30] that the homology $H_*(X) := H_*(X; \hat{\mathbb{Z}})$ of a profinite space X is defined to be the homology of the complex $C_*(X)$ consisting in degree n of the profinite groups $C_n(X) := \hat{F}_{\mathrm{ab}}(X_n)$, the free abelian profinite group on the profinite set X_n. The differentials d are the alternating sums $\sum_{i=0}^n d_i$ of the face maps d_i of X. If M is a profinite abelian group, then $H_*(X; M)$ is defined to be the homology of the complex $C_*(X; M) := C_*(X) \hat{\otimes} M$, where $\hat{\otimes}$ denotes the completed tensor product, see e.g. [34] §5.5.

For $X \in \hat{\mathcal{S}}_G$, the homotopy orbit space $X_{hG} = EG \times_G X$ can be viewed as the homotopy colimit of the G-action on X. Moreover, the homology $H_s(X; M)$ is itself a profinite G-module for any profinite abelian group M. This gives rise to the following spectral sequence.

Theorem 2.25. *Let X be a profinite G-space and M a profinite abelian group. There is a first quadrant homology spectral sequence for the homology groups of X_{hG} starting from the continuous homology $H_t(X; M)$ of G with coefficients in the profinite G-modules converging to the homology of the homotopy orbit space of X:*

$$E_{s,t}^2 = H_s(G, H_t(X; M)) \Rightarrow H_{s+t}(X_{hG}; M).$$

Proof. This is a profinite version of the homotopy colimit spectral sequence of Bousfield and Kan [3], XII §5.7. We can assume that X is fibrant in $\hat{\mathcal{S}}_G$. By [3], XII §5.2, in order to calculate the homotopy colimit in $\hat{\mathcal{S}}$ of the diagram induced by the G-action, one can first take a simplicial resolution of this diagram. In our case

this yields a simplicial profinite space $X \times G^*$, where, for every k, G^k denotes the constant simplicial set of the k-fold product of G. The homotopy colimit is then equal to the diagonal of the bisimplicial resolution of X induced by G, i.e.

$$X_{hG} \cong \mathrm{diag}(X \times G^*) \in \hat{\mathcal{S}}.$$

It follows immediately that, by applying homology, the bisimplicial profinite set yields a bisimplicial abelian group which has a profinite structure in each bilevel and in which the maps are continuous group homomorphisms. It is a standard argument to deduce from the bisimplicial abelian group a spectral sequence

$$E^2_{s,t} = \underset{G}{\mathrm{colim}}^s H_t(X; M) \Rightarrow H_{s+t}(\underset{G}{\mathrm{hocolim}}\, X; M)$$

where colim^s_G denotes the sth left derived functor of the functor induced by the G-action. It remains to remark that, all groups being equipped with a natural profinite structure, colim^s_G is the derived functor of colim_G in the category of profinite G-modules; and that $\mathrm{colim}_G B$ is the orbit group B/G of a profinite G-module B. Moreover, $H_s(G, B)$ is the sth left derived functor of the functor $B \mapsto B/G$ by [34], Proposition 6.3.4. $\qquad\square$

2.4. **Profinite G-spectra.** A profinite spectrum X consists of a sequence $X_n \in \hat{\mathcal{S}}_*$ of pointed profinite spaces for $n \geq 0$ and maps $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ in $\hat{\mathcal{S}}_*$. A morphism $f : X \to Y$ of spectra consists of maps $f_n : X_n \to Y_n$ in $\hat{\mathcal{S}}_*$ for $n \geq 0$ such that $\sigma_n(1 \wedge f_n) = f_{n+1}\sigma_n$. We denote by $\mathrm{Sp}(\hat{\mathcal{S}}_*)$ the corresponding category of profinite spectra. By Theorem 2.36 of [30], there is a stable homotopy category $\hat{\mathcal{SH}}$ of profinite spectra. In this model structure, a map $f : X \to Y$ is a stable equivalence if it induces a weak equivalence of mapping spaces $\mathrm{map}(Y, E) \to \mathrm{map}(X, E)$ for all Ω-spectra E; and f is a cofibration if $X_0 \to Y_0$ and the induced maps $X_n \amalg_{S^1 \wedge X_{n-1}} S^1 \wedge Y_{n-1} \to Y_n$ are monomorphisms for all n.

Now let G be as always a profinite group. We consider the simplicial finite set S^1 as a profinite G-space with trivial action.

Definition 2.26. *We call X a profinite G-spectrum if, for $n \geq 0$, each X_n is a pointed profinite G-space and each $S^1 \wedge X_n \to X_{n+1}$ is a G-equivariant map. We denote the category of profinite G-spectra by $\mathrm{Sp}(\hat{\mathcal{S}}_{*,G})$.*

Theorem 2.27. *There is a model structure on profinite G-spectra such that a map is a stable weak equivalence (resp. cofibration) if and only if it is a stable weak equivalence (resp. cofibration) in $\mathrm{Sp}(\hat{\mathcal{S}}_*)$. The fibrations are the maps with the right lifting property with respect to maps that are weak equivalences and cofibrations. We denote its homotopy category by $\hat{\mathcal{SH}}_G$.*

Proof. Starting with the model structure on $\hat{\mathcal{S}}_G$ of Theorem 2.19, the stable model structure is obtained in the same way as for $\hat{\mathcal{S}}$ from the techniques of [17] and the localization results of [29], Theorems 6 and 14, for fibrantly generated model categories. It is also clear from this construction and Theorem 2.19 that a map in $\mathrm{Sp}(\hat{\mathcal{S}}_{*,G})$ is a stable weak equivalence (resp. cofibration) if and only if it is a stable weak equivalence (resp. cofibration) in $\mathrm{Sp}(\hat{\mathcal{S}}_*)$. $\qquad\square$

Corollary 2.28. *If X is a profinite G-spectrum then each stable profinite homotopy group $\pi_k X$ is a profinite G-module.*

Corollary 2.29. *If a map f is a fibration in $\mathrm{Sp}(\hat{\mathcal{S}}_{*,G})$, then its underlying map is a fibration in $\mathrm{Sp}(\hat{\mathcal{S}}_*)$.*

Let $t : \mathrm{Sp}(\hat{\mathcal{S}}_*) \to \mathrm{Sp}(\hat{\mathcal{S}}_{*,G})$ be the functor that equips a spectrum X with the trivial G-action. Its right adjoint is the level-wise fixed point functor $(-)^G : \mathrm{Sp}(\hat{\mathcal{S}}_{*,G}) \to \mathrm{Sp}(\hat{\mathcal{S}}_*)$. Since t clearly preserves weak equivalences and cofibrations, we get the following statement as in [6].

Corollary 2.30. *The pair $(t, (-)^G)$ forms a Quillen pair of functors.*

2.5. **Homotopy orbit spectra.** Our aim is to construct a spectral sequence as above that starts with the continuous homology of G with coefficients the profinite G-module $\pi_k X$ and that converges to the stable homotopy groups of the homotopy orbit spectrum $X_{hG} := EG_+ \wedge_G X$ of the G-action on X.

For this purpose, we consider the simplicial resolution of the diagram induced by the G-action on X. As in [3] XII, it is defined to be the simplicial profinite spectrum $X \wedge (G^*)_+$. Here we denote again, for every k, by $(G^k)_+$ the constant simplicial set of the k-fold product of G as above but with an additional basepoint and in level n, $(X \wedge (G^k)_+)_n := X_n \wedge (G^k)_+ \in \hat{\mathcal{S}}_*$. We may consider $X \wedge (G^*)_+$ either as a bisimplicial profinite set or as a simplicial profinite spectrum. Since the diagonal functor d from pointed bisimplicial profinite sets to pointed simplicial profinite sets commutes with smashing with S^1, we may apply the diagonal functor level-wise to get a spectrum $d(X \wedge (G^*)_+)$ with $d(X \wedge (G^*)_+)_n = \mathrm{diag}(X_n \wedge (G^*)_+)$, cf. [20] 4.3. The homotopy colimit is then isomorphic to the diagonal spectrum $d(X \wedge (G^*)_+)$ of the simplicial resolution, i.e.

$$X_{hG} \cong d(X \wedge (G^*)_+) \in \mathrm{Sp}(\hat{\mathcal{S}}_*).$$

For a simplicial spectrum, Jardine shows in [20] §4, how to construct a spectral sequence that computes the homotopy groups of the diagonal spectrum, Corollary 4.22 of [20]:

$$(3) \qquad\qquad E^2_{s,t} = H_s(\pi_t(Y_*)) \Rightarrow \pi_{s+t}(d(Y)).$$

The whole construction can be applied in the category of simplicial profinite spectra. For a simplicial profinite spectrum $[k] \mapsto Y_k$, the stable homotopy group $\pi_t(Y_k)$ has a natural profinite structure and $\pi_t(Y_*)$ becomes a simplicial profinite abelian group. The resulting complex has continuous differentials. Its homology groups also carry an induced natural profinite structure. Hence (3) may be viewed as a spectral sequence in profinite abelian groups.

We use (3) for two applications. The first one shows that the diagonal functor from simplicial profinite spectra to profinite spectra and hence the homotopy orbit functor respects weak equivalences.

Proposition 2.31. *Let $X \to Y$ be a map between simplicial profinite spectra such that, for each $n \geq 0$, the map $X_n \to Y_n$ is a stable equivalence in $\mathrm{Sp}(\hat{\mathcal{S}}_*)$. Then the induced map $d(X) \to d(Y)$ is a stable equivalence in $\mathrm{Sp}(\hat{\mathcal{S}}_*)$.*

The second application of (3) is what we were really looking for. Let X be a profinite G-spectrum. Then the homology of the simplicial profinite abelian group $\pi_t(X \wedge (G^*)_+)$ is just the continuous group homology $H_s(G, \pi_t X)$ with profinite coefficients $\pi_t X$. Hence we get the following result.

Theorem 2.32. *Let G be a profinite group and let X be a profinite G-spectrum. Then there is a convergent spectral sequence*

$$E_{s,t}^2 = H_s(G; \pi_t(X)) \Rightarrow \pi_{s+t}(X_{hG}).$$

A spectral sequence for the homotopy orbit spectrum under an action of a profinite group G had already been studied in different contexts, in particular by Davis. In [6], Davis considers discrete G-spectra and calls a spectrum Y an f-spectrum if $\pi_q Y$ is a finite group for each integer q. Let G be a countably based profinite group and $Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \ldots$ a tower of G-f-spectra such that the level-wise taken homotopy limit $Y = \mathrm{holim}_i Y_i$ is a G-spectrum. Then the homotopy groups of Y are profinite groups. For this situation, Theorem 5.3 of [6] provides a spectral sequence as in Theorem 2.32. We remark that since $\pi_q Y_i$ is finite, for each Y_i there is some profinite G-spectrum X_i which is fibrant in $\mathrm{Sp}(\hat{\mathcal{S}}_*)$ such that its underlying spectrum, i.e. after forgetting the profinite structure, is weakly equivalent to Y_i. The homotopy limit X is then also weakly equivalent to Y and Y_{hG} is weakly equivalent to X_{hG}. Hence each tower Y of G-f-spectra may be considered as a profinite G-spectrum and the spectral sequence of Theorem 5.3 of [6] is a special case of the spectral sequence of Theorem 2.32 above that arises naturally in the category of profinite G-spectra for an arbitrary profinite group G.

3. Galois actions

Now we return to our motivating examples for profinite spaces and continuous group actions of the introduction. The starting point for étale homotopy theory is the work of Artin and Mazur [1]. The goal was to define invariants as in Algebraic Topology for a scheme X that depend only on the étale topology of X. They associated to a scheme X a pro-object in the homotopy category $\mathcal{H}$ of spaces. Friedlander rigidified the construction by associating to X a pro-object in the category $\mathcal{S}$ of simplicial sets. The construction is technical and we refer the reader to [11] for any details, in particular for the category of rigid hypercoverings. As a reminder for the reader who is familiar with the techniques, the definition is the following: For a locally noetherian scheme X, the étale topological type of X is the pro-simplicial set $\mathrm{Et}\, X := \mathrm{Re} \circ \pi : HRR(X) \to \mathcal{S}$ sending a rigid hypercovering $U.$ of X to the simplicial set of connected components of $U.$. If $f : X \to Y$ is a map of locally noetherian schemes, then the strict map $\mathrm{Et}\, f : \mathrm{Et}\, X \to \mathrm{Et}\, Y$ is given by the functor $f^* : HRR(Y) \to HRR(X)$ and the natural transformation $\mathrm{Et}\, X \circ f^* \to \mathrm{Et}\, Y$.

In [29] and [30], we studied a profinite version $\hat{\mathrm{Et}}$ of this functor by composing Et with the completion from pro-$\mathcal{S}$ to the category of simplicial profinite sets $\hat{\mathcal{S}}$. The advantage of $\hat{\mathrm{Et}}\, X$ is that we have taken the limit over all hypercoverings in a controlled way and obtain an actual simplicial set that still remembers the continuous invariants of X. Let us summarize the key properties of $\hat{\mathrm{Et}}\, X$ in the following proposition, which is due to Artin-Mazur [1] and Friedlander [11], but one might also want to have a look at [30] for the comparison with continuous cohomology of Jannsen [18].

Proposition 3.1. *1. Let $\bar{x}$ be a geometric point of X. It also determines a point in $\hat{\mathrm{Et}}\, X$. The profinite fundamental group $\pi_1(\hat{\mathrm{Et}}\, X, \bar{x})$ of $\hat{\mathrm{Et}}\, X$ as an object of $\hat{\mathcal{S}}$ is isomorphic to the étale fundamental group $\pi_1^{\mathrm{\acute{e}t}}(X, \bar{x})$ of X as a scheme.*

2. Let F be locally constant étale sheaf of profinite abelian groups on X. It corresponds bijectively to a local coefficient system F of profinite groups on $\hat{\mathrm{Et}}\, X$. Moreover, the cohomology of $\hat{\mathrm{Et}}\, X$ with profinite local coefficients in F equals the continuous étale cohomology of X, i.e. $H^*(\hat{\mathrm{Et}}\, X, F) \cong H^*_{\mathrm{cont}}(X, F)$.

Now let k be a field, $\bar{k}$ a separable closure of k and $G_k := \mathrm{Gal}(\bar{k}/k)$. Let X be a variety over k, i.e. a separated reduced and irreducible scheme of finite type over k, and let $\bar{X} = X \otimes_k \bar{k}$ be the base change of X to $\bar{k}$. Unfortunately, it is not clear if $\hat{\mathrm{Et}}\, \bar{X}$ is always a profinite G_k-space in the above sense. The action of G_k on the sets of connected components of rigid hypercovers might not be continuous in each level. Nevertheless, there is a canonical model for $\hat{\mathrm{Et}}\, \bar{X}$ in $\hat{\mathcal{S}}_G$.

Lemma 3.2. *The canonical G_k-equivariant map $\alpha : \hat{\mathrm{Et}}\, \bar{X} \longrightarrow \lim_L \hat{\mathrm{Et}}\, X_L$ is a weak equivalence, where the limit is taken over all finite Galois extensions L/k in $\bar{k}$.*

Proof. Let L/k be a finite Galois extension with Galois group G_L. Since fundamental groups of profinite spaces commute with limits, there is a canonical isomorphism

$$\pi_1(\lim_L \hat{\mathrm{Et}}\, X_L) \cong \lim_L \pi_1(\hat{\mathrm{Et}}\, X_L).$$

Moreover, by [14] IX, §6, we know that $\pi_1(\hat{\mathrm{Et}}\, \bar{X}) = \pi_1^{\mathrm{ét}}(\bar{X})$ is isomorphic to $\lim_L \pi_1(\hat{\mathrm{Et}}\, X_L)$. This shows that α induces isomorphisms on fundamental groups. It remains to show that α also induces isomorphisms on cohomology with local coefficients of finite abelian groups. This from the fact that $H^{ast}_{\mathrm{ét}}(\bar{X}; F)$ is equal to the colimit $\mathrm{colim}_L H^*_{\mathrm{ét}}(X_L; F_L)$ for any locally constant sheaf F on $\bar{X}$ whose pullback to X_L is denoted by F_L. From the analogous equality $H^*(\lim_L \hat{\mathrm{Et}}\, X_L; F) = \mathrm{colim}_L H^*(\hat{\mathrm{Et}}\, X_L; F)$ and Proposition 3.1, we deduce that α is a weak equivalence. $\square$

Since the action of G_k on $\hat{\mathrm{Et}}\, X_L$ factors through the finite group $\mathrm{Gal}(L/k)$, this action is continuous on the profinite space $\hat{\mathrm{Et}}\, X_L$. As G_k is the limit of all the $\mathrm{Gal}(L/k)$, this shows that the action of G_k on $\lim_L \hat{\mathrm{Et}}\, X_L$ is continuous, cf. [2] III §7, No 1. We will use this profinite G_k-space as a continuous model for $\hat{\mathrm{Et}}\, \bar{X}$ in $\hat{\mathcal{S}}_G$ and will denote it by

$$\mathrm{c}\hat{\mathrm{Et}}\, \bar{X} := \lim_L \hat{\mathrm{Et}}\, X_L.$$

Remark 3.3. This problem vanishes if G_k is strongly complete, i.e. if it is isomorphic to its profinite completion, or in other words, if every subgroup of finite index is open, see [34]. In this case, the G_k-action on $\hat{\mathrm{Et}}\, \bar{X}$ would be continuous for any variety X. The class of strongly complete profinite groups contains the class of all finitely generated profinite group by the work of Nikolov and Segal [26]. For example the absolute Galois group of p-adic local fields are finitely generated, cf. [19]. The absolute Galois group of a number field is in general not strongly complete as subgroups of finite index which are not open be easily constructed in such groups.

By [11] and [30], we know that $\hat{\mathrm{Et}}\, k$ is homotopy equivalent to BG_k and $\hat{\mathrm{Et}}\, \bar{k}$ to EG_k in $\hat{\mathcal{S}}$. As mentioned in the introduction the natural sequence (1) inspires us

to think of $\hat{\mathrm{Et}}\, X$ as the homotopy orbit space of $\hat{\mathrm{Et}}\, \bar{X}$, just as Cox showed for real algebraic varieties in [5] Theorem 1.1. The following theorem generalizes Cox's result to arbitrary fields. The point is that $\hat{\mathrm{Et}}\, \bar{X} \to \hat{\mathrm{Et}}\, X$ is homotopy equivalent to a principal G_k-fibration, see [30], p. 593.[1]

Theorem 3.4. *Let k be a field with absolute Galois group G_k and let X be a geometrically connected variety over k. Then the canonical map*

$$\varphi : \mathrm{c}\hat{\mathrm{Et}}\, \bar{X} \times_{G_k} EG_k \to \hat{\mathrm{Et}}\, X$$

is a weak equivalence of profinite spaces.

Proof. By the definition of weak equivalences, we have to show that φ induces an isomorphism on the profinite fundamental groups and on continuous cohomology with finite abelian coefficients systems. Let us start with the fundamental groups. We know from the work of Grothendieck [14] IX, Théorème 6.1, that there is a short exact sequence

$$1 \longrightarrow \pi_1^{\text{ét}}(\bar{X}, \bar{x}) \longrightarrow \pi_1^{\text{ét}}(X, x) \to G_k \longrightarrow 1$$

for every geometric point $\bar{x}$ of $\bar{X}$ with image x in X. On the other hand, the map of profinite spaces $\mathrm{c}\hat{\mathrm{Et}}\, \bar{X} \times EG_k \to \mathrm{c}\hat{\mathrm{Et}}\, \bar{X} \times_{G_k} EG_k$ is a principal G_k-fibration by definition, see [30]. Hence it is also locally trivial, see e.g. [13] V, Lemma 2.5, and may be considered as a Galois covering with group G_k. By the classification of coverings of profinite spaces via the fundamental group in [30], Corollary 2.3, we deduce that there is a similar short exact sequence for profinite spaces such that $\pi_1(\varphi)$ fits in a commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\mathrm{c}\hat{\mathrm{Et}}\, \bar{X} \times EG_k, \bar{x}) & \longrightarrow & \pi_1(\mathrm{c}\hat{\mathrm{Et}}\, \bar{X} \times_{G_k} EG_k, x) & \longrightarrow & G_k & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \pi_1^{\text{ét}}(\bar{X}, \bar{x}) & \longrightarrow & \pi_1^{\text{ét}}(X, x) & \longrightarrow & G_k & \longrightarrow & 1
\end{array}
$$

for every basepoint $\bar{x}$ of $\bar{X}$. Since EG_k is contractible, the left vertical arrow is an isomorphism and we conclude that φ induces an isomorphism on fundamental groups.

To prove that φ also induces an isomorphism on cohomology we apply two Serre spectral sequences. Let F be a locally constant étale sheaf of finite abelian groups on X. On the one hand there is the Hochschild-Serre spectral sequence for étale cohomology starting from continuous cohomology of G_k with coefficients in the discrete G_k-module $H^t_{\text{ét}}(\bar{X}; F)$, cf. [22]:

$$E_2^{s,t} = H^s(G_k; H^t_{\text{ét}}(\bar{X}; F)) \Rightarrow H^{s+t}_{\text{ét}}(X; F).$$

On the other hand the fibre sequence

$$\mathrm{c}\hat{\mathrm{Et}}\, \bar{X} \longrightarrow \mathrm{c}\hat{\mathrm{Et}}\, \bar{X} \times_{G_k} EG_k \longrightarrow BG_k$$

induces a Serre spectral sequence

$$E_2^{s,t} = H^s(G_k; H^t(\mathrm{c}\hat{\mathrm{Et}}\, \bar{X}; F)) \Rightarrow H^{s+t}(\mathrm{c}\hat{\mathrm{Et}}\, \bar{X} \times_{G_k} EG_k; F)$$

---

[1]In [30] and in a previous version of this paper, it was stated that this map is a principal fibration, which is only true up to homotopy. So one may find here a rigorous treatment of the problem.

where F also denotes the associated local coefficient system on $\hat{\mathrm{E}}\mathrm{t}\,\bar{X}$ by Proposition 3.1. This spectral sequence may be constructed in the profinite setting just as in [9], see also [25] §1.5 and [7] §1.5 for pro-p-versions. It remains to observe that there is a natural isomorphism between these spectral sequences which is compatible with φ using the isomorphism $H^t_{\text{ét}}(\bar{X}; F) \cong H^t(\hat{\mathrm{E}}\mathrm{t}\,\bar{X}; F)$. Since these groups vanish for $t > \dim \bar{X}$, the two spectral sequences are strongly convergent which finishes the proof of the theorem. $\square$

Hence we may consider $\hat{\mathrm{E}}\mathrm{t}\,X$ as the homotopy orbit space of $\hat{\mathrm{E}}\mathrm{t}\,X_{\bar{k}}$ under its natural Galois action. We will use this key theorem for three applications. On the one hand we deduce Galois descent spectral sequences for étale (co)homology theories. The last application is a remark on Grothendieck's section conjecture for smooth proper curves of genus at least two over number fields. But first we show that we can lift this equivalence of the two points of view to the level of motivic spectra.

We know that the étale realization functor above can be extended to a functor from motivic spectra [37] to the homotopy category of profinite spectra over $\hat{\mathrm{E}}\mathrm{t}\,k$, cf. [29], Theorem 31:
$$\hat{\mathrm{E}}\mathrm{t} : \mathcal{SH}(k) \to \hat{\mathcal{SH}}/\hat{\mathrm{E}}\mathrm{t}\,k.$$
This extension can be achieved with the model structure on $\hat{\mathcal{S}}$ and $\mathrm{Sp}(\hat{\mathcal{S}}_*)$ of Theorem 2.3 and Theorem 2.36 of [30], respectively, if char $k = 0$. If char $k = p > 0$, we have to complete away from the characteristic by using the $\mathbb{Z}/\ell$-model structure on $\hat{\mathcal{S}}$ and $\mathrm{Sp}(\hat{\mathcal{S}}_*)$ for any prime $\ell \neq p$ of [25], see also [29] and Remark 2.21 above. Now we remark that the adjointness discussed in the beginning of Section 2.3 of taking homotopy orbits and pullbacks via maps to BG_k has an analogue for profinite spectra. This implies that we can reconstruct étale realization as a functor to the category of profinite G_k-spectra. Hence the following theorem is in this sense equivalent to Theorem 31 of [29].

Theorem 3.5. *Let k be a field of characteristic zero and let G_k be its absolute Galois group. The étale realization functor above defines a functor*
$$\hat{\mathrm{E}}\mathrm{t} : \mathcal{SH}(k) \to \hat{\mathcal{SH}}_{G_k}$$
to the stable homotopy category of profinite G_k-spectra by sending a motivic spectrum E to $\hat{\mathrm{E}}\mathrm{t}\,E_{\bar{k}}$, where $E_{\bar{k}}$ is the base change of E to $\bar{k}$, with its natural G_k-action. If k has positive characteristic p, the same statement holds when we equip $\hat{\mathcal{S}}$ and $\mathrm{Sp}(\hat{\mathcal{S}}_*)$ with the $\mathbb{Z}/\ell$-model for any prime $\ell \neq p$.

3.1. **Galois descent.** As an application of the homotopy orbit spectral sequence we consider a variant of étale homotopy groups. For a pointed locally noetherian scheme, we define $\pi^{\text{ét},s}_* X := \pi^s_*(\Sigma^\infty \hat{\mathrm{E}}\mathrm{t}\,X)$ to be the stable étale homotopy groups of X. The absolute Galois group G_k acts continuously on each profinite group $\pi^{\text{ét},s}_q(\bar{X})$. There is the following Galois descent spectral sequence for these groups by Theorem 2.32.

Theorem 3.6. *Let X be a geometrically connected variety over a field k with absolute Galois group G_k. There is a convergent spectral sequence for the stable étale homotopy groups of X:*
$$E^2_{p,q} = H_p(G_k; \pi^{\text{ét},s}_q(\bar{X})) \Rightarrow \pi^{\text{ét},s}_{p+q}(X).$$

In the same way we get Galois descent spectral sequences for étale topological cohomology theories, e.g. étale cobordism [29]. Let MU be the simplicial spectrum representing topological complex cobordism and let $\hat{M}U$ be its profinite completion. In [29], an étale topological version of cobordism for smooth schemes has been studied. It is the theory represented by $\hat{M}U$ via $\hat{\text{Et}}$, i.e. in degree n we set

$$\hat{M}U_{\text{ét}}^n(X) := \text{Hom}_{\hat{\mathcal{SH}}}(\Sigma^\infty \hat{\text{Et}}\, X, \hat{M}U[n])$$

where $\hat{M}U[n]$ denotes the nth shift of $\hat{M}U$ and X is assumed to be a pointed scheme. We can reformulate this definition using function spectra and get an isomorphism

(4) $$\hat{M}U_{\text{ét}}^n(X) \cong \pi_n \text{hom}_{\text{Sp}(\hat{\mathcal{S}}_*)}(\Sigma^\infty \hat{\text{Et}}\, X, \hat{M}U).$$

Let us denote the function spectrum on the right hand side of (4) by

$$\hat{M}U_{\text{ét}}^X := \text{hom}_{\hat{\mathcal{SH}}/\hat{\text{Et}}\, k}(\Sigma^\infty \hat{\text{Et}}\, X, R\hat{M}U)$$

where R means a fibrant replacement in $\text{Sp}(\hat{\mathcal{S}}_*)$. Now this description and an analogue of Proposition 3.4 implies that étale cobordism satisfies Galois descent in the following sense, generalizing [10], Proposition 7.1. Therefore, we start with the following lemma.

Lemma 3.7. *For each finite Galois extension L/k with Galois group $G_L = \text{Gal}(L/k)$ and $X_L = X \otimes_k L$, there is a natural equivalence of simplicial spectra*

$$\hat{M}U_{\text{ét}}^X \xrightarrow{\simeq} (\hat{M}U_{\text{ét}}^{X_L})^{hG_L}$$

where G_L acts on $\hat{M}U_{\text{ét}}^{X_L}$ via its induced action on $\hat{\text{Et}}\, X_L$.

Proof. The assertion is implied by the following sequence of equivalences, where we omit Σ^∞:

$$\begin{aligned}
\text{hom}_{\text{Sp}(\hat{\mathcal{S}}_*)}(\hat{\text{Et}}\, X, R\hat{M}U) \quad &\xrightarrow{\simeq} \quad \text{hom}_{\text{Sp}(\hat{\mathcal{S}}_*)}(\hat{\text{Et}}\, X_L \times_{G_L} EG_L, R\hat{M}U) \\
&\xrightarrow{\simeq} \quad \text{hom}_{\text{Sp}(\mathcal{S}_{*,G})}(EG_L, \text{hom}(\hat{\text{Et}}\, X_L, R\hat{M}U)) \\
&\xrightarrow{\simeq} \quad (\hat{M}U_{\text{ét}}^X)^{hG_L}
\end{aligned}$$

where the first equivalence follows from Theorem 3.4, the second follows from adjointness for the simplicial finite set EG_L and the third one is the definition of homotopy fixed point spectra for finite groups acting on simplicial spectra. $\square$

Theorem 3.8. *Let k be a field with absolute Galois group G_k and let X be a geometrically connected pointed variety over k. There is a spectral sequence*

$$E_2^{s,t} = H^s(G_k; \hat{M}U_{\text{ét}}^t(\bar{X})) \Rightarrow \hat{M}U_{\text{ét}}^{s+t}(X)$$

starting from continuous cohomology of G_k with coefficients the discrete G_k-module $\hat{M}U_{\text{ét}}^*(\bar{X})$. This spectral sequence converges if G_k has finite cohomological dimension.

Proof. Each finite quotient G_L of G_k induces a finite Galois covering $X_L \to X$ which is homotopy equivalent to finite Galois covering of the profinite space $\hat{\text{Et}}\, X$ using the argument in the proofs of Theorem 3.4 and Lemma 3.2. The well-known homotopy fixed point spectral sequence for finite groups acting on simplicial spectra together with the Lemma 3.7 yield a spectral sequence

$$E_2^{s,t} = H^s(G_L; \hat{M}U_{\text{ét}}^t(X_L)) \Rightarrow \hat{M}U_{\text{ét}}^{s+t}(X)$$

for every i. Now the weak equivalence $\hat{\mathrm{Et}}\,\bar{X} \simeq \lim_L \hat{\mathrm{Et}}\,X_L$ of Lemma 3.2 implies $\hat{MU}^t_{\text{ét}}(\bar{X}) \cong \operatorname{colim}_L \hat{MU}^t_{\text{ét}}(X_L)$ and hence there is an isomorphism

$$H^s(G_k; \hat{MU}^t_{\text{ét}}(\bar{X})) \cong \operatorname*{colim}_L H^s(G_L; \hat{MU}^t_{\text{ét}}(X_L)).$$

Since spectral sequences commute with colimits, this implies the assertion of the theorem. $\qquad\square$

The homological counter part, called étale bordism, is defined as

$$\hat{MU}^{\text{ét}}_n(X) := \operatorname{Hom}_{\hat{\mathcal{SH}}}(S^n, \Sigma^\infty \hat{\mathrm{Et}}\,X \wedge \hat{MU}).$$

In this case, the descent spectral sequence for étale bordism has a more direct construction as the homotopy orbit spectral sequence of a generalized homology theory as in Theorem 2.25 above.

Theorem 3.9. *Let k be a field with absolute Galois group G_k and let X be a geometrically connected pointed variety over k. There is a convergent spectral sequence for the étale bordism of X:*

$$E^2_{s,t} = H_s(G_k; \hat{MU}^{\text{ét}}_t(\bar{X})) \Rightarrow \hat{MU}^{\text{ét}}_{s+t}(X).$$

3.2. **A remark on Grothendieck's section conjecture.** We conclude with an application of the developed theory of homotopy fixed points to Galois actions.[2] Let us briefly recall the statement of Grothendieck's section conjecture [15]. It is part of a much more general picture drawn by Grothendieck in [15] which predicts that, for some class of varieties over k, the functor of taking fundamental groups should be in some sense fully faithful. Detailed accounts on the conjecture can be found e.g. in [23], [21] and [35]. Let k be a field and G_k its absolute Galois group. Let X be a geometrically connected variety over k. We have already used that the functoriality of $\pi_1 = \pi_1^{\text{ét}}$ induces a short exact sequence

$$1 \longrightarrow \pi_1\bar{X} \longrightarrow \pi_1 X \longrightarrow G_k \longrightarrow 1$$

where we omit the basepoints for this discussion. Another application of the functoriality of π_1 shows that every k-rational point $a \in X(k)$ induces a section $s_a : G_k \to \pi_1 X$ which is well-defined up to conjugacy by $\pi_1\bar{X}$. The section conjecture of Grothendieck's in [15] predicts that this map has an inverse.

Conjecture 3.10. (Grothendieck) *Let k be a field which is finitely generated over $\mathbb{Q}$ and let X be a smooth, projective curve of genus at least two. The map $a \mapsto s_a$ is a bijection between the set $X(k)$ of k-rational points of X and the set of $\pi_1\bar{X}$-conjugacy classes of sections $G_k \to \pi_1 X$.*

It is well-known that the map $a \mapsto s_a$ is injective. The hard part is the surjectivity. The varieties that the Grothendieck conjecture is about, especially the curves of Conjecture 3.10, are so called $K(\pi, 1)$-varieties. And here is the point where étale homotopy enters the stage. In terms of étale homotopy theory a variety X is a $K(\pi, 1)$-variety if the profinite universal covering space of $\hat{\mathrm{Et}}\,X$ is contractible. Or in other words, $\hat{\mathrm{Et}}\,X$ is weakly equivalent in $\hat{\mathcal{S}}$ to the profinite classifying space $B\pi_1 X$. Just as for spaces, it is also known for profinite spaces that there is a

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[2]I would like to thank Kirsten Wickelgren for interesting discussions about this topic.

bijection between the set of homotopy classes of continuous maps of Eilenberg-MacLane spaces $\mathrm{Hom}_{\hat{\mathcal{H}}}(K(G,1) \to K(\pi,1))$ and the set of outer continuous group homomorphisms $\mathrm{Hom}_{\mathrm{out}}(G,\pi)$. In light of the previous discussion this shows there is a bijection

(5) $$\mathrm{Hom}_{\hat{\mathcal{H}}/\hat{\mathrm{E}}\mathrm{t}\,k}(\hat{\mathrm{E}}\mathrm{t}\,k, \hat{\mathrm{E}}\mathrm{t}\,X) \cong \mathrm{Hom}_{\mathrm{out},G_k}(G_k, \pi_1 X),$$

where the right hand side denotes outer homomorphisms that are compatible with the projection to G_k. So Conjecture 3.10 may be restated in the way that $\hat{\mathrm{E}}\mathrm{t}$ is a fully faithful functor from k-rational points to homotopy classes of maps from $\hat{\mathrm{E}}\mathrm{t}\,k$ to $\hat{\mathrm{E}}\mathrm{t}\,X$.

All this is of course just a reformulation. But the point we want to stress is that the machinery of Galois actions developed above provides an interesting point of view for Conjecture 3.10. Namely, it is an elementary fact that there is a bijection between the set of $\pi_1\bar{X}$-conjugacy classes of sections $G_k \to \pi_1 X$ and the set of (non-abelian) continuous cohomology $H^1(G_k; \pi_1\bar{X})$. By Proposition 2.23, $H^1(G_k; \pi_1\bar{X})$ is isomorphic to $\pi_0 \hom_{G_k}(EG_k, B\pi_1\bar{X}) = \pi_0(B\pi_1\bar{X})^{hG_k}$. Moreover, as X is a $K(\pi,1)$-variety, so is $\bar{X}$, i.e. $\hat{\mathrm{E}}\mathrm{t}\,\bar{X}$ is weakly equivalent to $B\pi_1\bar{X}$. This yields a natural bijection

$$H^1(G_k; \pi_1\bar{X}) \cong \pi_0(\mathrm{c}\hat{\mathrm{E}}\mathrm{t}\,\bar{X})^{hG_k}.$$

Via Theorem 3.4 this is an immediate reformulation of (5). Nevertheless, the slight shift of the point of view is very interesting, since the section conjecture over the real numbers $\mathbb{R}$ and topological analogues of it could be proved by Pal in [27] using fixed and homotopy fixed point methods for finite groups. Further studies in this direction have been done by Pal in [28]. The new input of this paper consists in the rigorous framework for homotopy fixed points for étale homotopy types for varieties over any base field which had been missing so far.

What one would like to do now is to factor the map

(6) $$X(k) \to \pi_0(\mathrm{c}\hat{\mathrm{E}}\mathrm{t}\,\bar{X})^{hG_k}$$

through some fixed point set under the G_k-action. Then the geometric part of the problem would be to show that $X(k)$ is isomorphic to this fixed point set. The homotopy theoretical part would be to show that the fixed point set is isomorphic to the homotopy fixed point set in the right hand side of (6). This might be possible by transferring comparison results for finite groups to the case of the special profinite groups G_k acting on the profinite classifying space $B\pi_1\bar{X} \simeq \mathrm{c}\hat{\mathrm{E}}\mathrm{t}\,\bar{X}$.

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